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Existence and Iteration of Monotone Positive Solutions for Third-order BVP with Integral Boundary Conditions

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Abstract

This paper is concerned with the third-order boundary value problem with integral boundary conditions.By applying iterative techniques, we obtain the existence of monotone positive solutions.

Keywords

Third-order BVP; Iterative method; Positive solutions.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics,e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [9].

Third-order two-point or multi-point boundary value problems (BVPs for short) have attracted a lot of attention [2, 5, 10–16, 18–21]. For example, in 2008, Sun [18] studied the existence of nondecreasing positive solutions for the nonlinear third-order two point boundary value problem

$$\begin{cases} u'''(t)+q(t)f(t,u(t),u'(t))=0, 0<t<1, \\ u(0)=u''(0)=u'(1)=0. \end{cases}$$

The iterative schemes for approximating the solutions are obtained by applying a monotone iterative method. On the other hand, BVPs with integral boundary conditions have been used in the description of many phenomena in the applied sciences. It cover multi-point BVPs as special cases. Recently, third-order BVPs with integral boundary conditions have attracted the attention of some authors. we refer the readers to [3, 4, 6–8, 17, 22, 23] and the references therein. It is worth mentioned that, in 2009, Boucherif, Bouguima, Al-Malki and Benbouziane [4] studied the nonlinear third order differential equations with integral boundary conditions

$$\begin{cases} y'''(t)=f(t,y(t),y'(t),y''(t)), 0<t<1, \\ y(0)=0, \\ y'(0)-ay''(0)=\int_0^1 h_1(y(s),y'(s))ds, \\ y'(1)+by''(1)=\int_0^1 h_2(y(s),y'(s))ds. \end{cases}$$

They provided sufficient conditions on the nonlinearity and the functions appearing in the boundary conditions that guarantee the existence of at least one solution to the problem. Their technique is based on a priori bounds and fixed point theorems. In 2010, Sun and Li [17] concerned with the following third-order boundary value problem with integral boundary conditions

$$\begin{cases} u'''(t)+f(t,u(t),u'(t))=0, t\in[0,1], \\ u(0)=u'(0)=0, u'(1)=\int_0^1 g(t)u'(t)dt. \end{cases}$$

By using the Guo-Krasnoselskii fixed point theorem, some sufficient conditions are obtained for the existence and nonexistence of monotone positive solution to the above problem.

In this paper, we are concerned with the following third-order BVP with integral boundary conditions

$$\begin{cases} u'''(t)+a(t)f(t,u(t),u'(t))=0, t\in(0,1), \\ u(0)=u''(0)=0, u'(1)=\int_0^1 g(t)u(t)dt. \end{cases} \tag{P}$$

By applying iterative methods, we not only obtain the existence of monotone positive solutions, but also establish iterative schemes for approximating the solutions. Here, monotone positive solutions mean nondecreasing, nonnegative and nontrivial solutions. We do not rely on the Guo-Krasnoselskii fixed point theorem and the technique which is based on a priori bounds and fixed point theorems. Our main tool is the following theorem.

Theorem 1: [1] Let K be a normal cone of a Banach space E and $v_0 \le w_0$.

Suppose that

(a_1) $T:[v_0,w_0]\to E$ is completely continuous;

(a_2) T is monotone increasing on $[v_0,w_0]$;

(a_3) v_0 is a lower solution of T, that is, $v_0 \le Tv_0$;

(a_4) w_0 is an upper solution of T, that is, $Tw_0 \le w_0$. Then the iterative sequences

$v_n = Tv_{n-1}$ and $w_n = Tw_{n-1}$ ($n = 1, 2, 3, \cdots$)

satisfy

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0$$

$\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ converge to v , $w\in[v_0,w_0]$ which are fixed points of T .respectively.

Throughout this paper, we always assume that a, f, g satisfy:

(A_1) $a\in C([0,1]\times[0,+\infty))$ and a is not identically zero,

(A_2) $f\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$

(A_3) $g\in C([0,1]\times[0,+\infty))$.

2. Preliminaries

For convenience, we denote $\mu=\int_0^1 tg(t)dt$.

Lemma 1: *Let* $\mu\ne 1$.. Then for any $h\in C[0,1]$, the BVP

$$\begin{cases} -u'''(t)=h(t), t\in[0,1], \\ u(0)=u''(0)=0, u'(1)=\int_0^1 g(t)u(t)dt. \end{cases} \tag{1}$$

has a unique solution

$$u(t)=\int_0^1[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau]h(s)ds, t\in[0,1], \tag{2}$$

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Where

$$G(t,s)=\begin{cases} t(1-s), 0\le t\le s\le 1, \\ t(1-s)-\dfrac{(t-s)^2}{2}, 0\le s\le t\le 1. \end{cases}$$

Proof. Let u be a solution of the BVP (1). Then we may suppose that

$$u(t)=\int_0^1 G(t,s)h(s)ds+At^2+Bt+C, t\in[0,1]$$

By the boundary conditions in formula (1), we have

$$A=C=0 \text{ and } B=\frac{1}{1-\mu}\int_0^1 h(s)\int_0^1 G(\tau,s)g(\tau)d\tau ds.$$

Therefore, the BVP (1) has a unique solution

$$u(t)=\int_0^1 [G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau]h(s)ds, t\in[0,1].$$

Lemma 2: [18] For any $(t,s)\in[0,1]\times[0,1]$,

$$0\le\frac{\partial G(t,s)}{\partial t}\le 1-s,\ (t-\frac{t^2}{2})(1-s)\le G(t,s)\le t(1-s).$$

3. Main Results

In the remainder of this paper, we always assume that $\mu<1$. If we denote

$$\Lambda=\frac{1-\mu}{\int_0^1(1-s)a(s)ds}, \text{ then } \Lambda>0.$$

Theorem 2: Assume that $f(t,0,0)$ is not identically zero on $[0,1]$ *and there exists a constant* $R>0$ such that

$$f(t,u_1,v_1)\le f(t,u_2,v_2)\le\Lambda R, 0\le t\le 1, 0\le u_1\le u_2\le R, 0\le v_1\le v_2\le R \quad (3)$$

then the BVP (1) has monotone positive solutions.

Proof. Let $C^1[0,1]$ be equipped with the norm $\|u\|=\max\left\{\max_{t\in[0,1]}|u(t)|, \max_{t\in[0,1]}|u'(t)|\right\}$

And

$$K=\{u\in E:u(t)\geq 0 \ and\ u'(t)\geq 0\ for\ t\in[0,1]\}.$$

Then K is a normal cone in Banach space E. Now, we define an operator $T:K\to E$ by

$$(Tu)(t)=\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u(s),u'(s))ds, t\in[0,1],$$

Then

$$(Tu)'(t)=\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u(s),u'(s))ds, t\in[0,1],$$

which together with(A_1)(A_2)(A_3)and Lemmas (2) implies that $T:K\to K$. Obviously, the fixed points of T are the monotone nonnegative solutions of the BVP (P).

Let $v_0(t)=0$, $w_0(t)=Rt$, $t\in[0,1]$. We divide our proof into the following steps:

Step 1. We verify that $T:[v_0,w_0]\to K$ is completely continuous.

First, we prove that T is a compact operator. Let D is a bounded set in $[v_0,w_0]$. We will prove that $T(D)$ is relatively compact in K.

For any $\{y_k(t)\}_{k=1}^{\infty}\subset T(D)$,there exist $\{x_k\}_{k=1}^{\infty}\subset D$ such that $y_k=Tx_k$. Obviously, $0\leq x_k(t)\leq R$ and $0\leq x_k'(t)\leq R$ for $t\in[0,1]$. Then for any k, by Lemma 2 and formula (3) we have $|y_k(t)|=|(Tx_k)(t)|$.

$$=\left|\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,x_k(s),x_k'(s))ds\right|$$
$$\leq \Lambda R\int_0^1\left[t(1-s)+\frac{t(1-s)}{1-\mu}\int_0^1 \tau g(\tau)d\tau\right]a(s)ds$$
$$=\frac{\Lambda Rt}{1-\mu}\int_0^1(1-s)a(s)ds\leq R, t\in[0,1]$$

which implies that $\{y_k\}_{k=1}^{\infty}$ is uniformly bounded. Similarly, for any k, we have

$$|y_k'(t)|=|(Tx_k)'(t)|$$
$$=\left|\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,x_k(s),x_k'(s))ds\right|$$
$$\leq \Lambda R\int_0^1\left[(1-s)+\frac{1-s}{1-\mu}\int_0^1 \tau g(\tau)d\tau\right]a(s)ds$$
$$=\frac{\Lambda R}{1-\mu}\int_0^1(1-s)a(s)ds\leq R, t\in[0,1]$$

which shows that $\{y_k'\}_{k=1}^{\infty}$ is also uniformly bounded. This indicates that $\{y_k\}_{k=1}^{\infty}$ is equicontinuous. It follows from Arzela-Ascoli theorem that $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Without loss of generality, we may assume that converges in $C[0,1]$.On the other hand, by the uniform continuity of $\frac{\partial G(t,s)}{\partial t}$, we know that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| < \frac{\varepsilon}{\Lambda R \int_0^1 a(s)ds}, s \in [0,1].$$

Then for any $k, t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} &|y_k'(t_1) - y_k'(t_2)| = |(Tx_k)'(t_1) - (Tx_k)'(t_1)| \\ &= \left|\int_0^1 \left(\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right) a(s) f(s, x_k(s), x_k'(s))ds\right| \\ &\leq \int_0^1 \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| a(s) f(s, x_k(s), x_k'(s))ds < \varepsilon, \end{aligned}$$

which implies that $\{y_k'\}_{k=1}^{\infty}$ is equicontinuous. Again, by Arzela-Ascoli theorem, we know that $\{y_k'\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$.

Therefore, $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C^1[0,1]$. Thus, we have shown that T is a compact operator. Next, we prove that $T:[v_0, w_0] \to K$ is continuous.

Suppose that $u_m, u \in [v_0, w_0]$ and $\|u_m - u\| \to 0 (m \to \infty)$. Then for any m and $t \in [0,1]$, in view of Lemma 2 and formula (3), we have

$$\left[G(t,s) + \frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right] a(s) f(s, u_m(s), u_m'(s)) \leq \frac{(1-s)a(s)\Lambda R}{1-\mu},\ (t,s) \in [0,1] \times [0,1]$$

And

$$\left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right] a(s) f(s, u_m(s), u_m'(s)) \leq \frac{(1-s)a(s)\Lambda R}{1-\mu},\ (t,s) \in [0,1] \times [0,1].$$

By applying Lebesgue Dominated Convergence theorem, we get that

$$\lim_{m\to\infty}(Tu_m)(t) = \lim_{m\to\infty}\int_0^1 \left[G(t,s) + \frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right] a(s) f(s, u_m(s), u_m'(s))ds$$

$$=\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u(s),u'(s))ds$$
$$=(Tu)(t), t\in[0,1]$$

And

$$\lim_{m\to\infty}(Tu_m)'(t)=\lim_{m\to\infty}\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u_m(s),u_m'(s))ds$$
$$=\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u(s),u'(s))ds$$
$$=(Tu)'(t), t\in[0,1]$$

which indicates that $T:[v_0,w_0]\to K$ is continuous. Therefore, $T:[v_0,w_0]\to K$ is completely continuous.

Step 2. We assert that T is monotone increasing on $[v_0,w_0]$.

Suppose that $u,v\in[v_0,w_0]$ and $u\le v$. Then $0\le u(t)\le v(t)\le R$ and $0\le u'(t)\le v'(t)\le R$, for $t\in[0,1]$. By formula (3), we have

$$(Tu)(t)=\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u(s),u'(s))ds$$
$$\le\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,v(s),v'(s))ds$$
$$=(Tv)(t), t\in[0,1]$$

And

$$(Tu)'(t)=\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,u(s),u'(s))ds$$
$$\le\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,v(s),v'(s))ds$$
$$=(Tv)'(t), t\in[0,1]$$

which shows that $Tu\le Tv$.

Step 3. We prove that v_0 is a lower solution of T.

For any $t\in[0,1]$, we know that

$$(Tv_0)(t)=\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,0,0)ds\ge 0=v_0(t)$$

and

$$(Tv_0)'(t)=\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,0,0)ds\geq 0=v_0'(t)$$

which implies that $v_0 \leq Tv_0$.

Step 4. We show that w_0 is an upper solution of T.

It follows from Lemma 2 and formula (3) that

$$\begin{aligned}(Tw_0)(t)&=\int_0^1\left[G(t,s)+\frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,w_0(s),w_0'(s))ds\\ &\leq\frac{\Lambda Rt}{1-\mu}\int_0^1(1-s)a(s)ds\\ &=w_0(t),t\in[0,1]\end{aligned}$$

And

$$\begin{aligned}(Tw_0)'(t)&=\int_0^1\left[\frac{\partial G(t,s)}{\partial t}+\frac{1}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau\right]a(s)f(s,w_0(s),w_0'(s))ds\\ &\leq\frac{\Lambda R}{1-\mu}\int_0^1(1-s)a(s)ds\\ &=w_0'(t),t\in[0,1]\end{aligned}$$

which indicates that $Tw_0 \leq w_0$.

Step 5. We prove that the BVP (P) has monotone positive solutions. In fact, if we construct sequences $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ as follows:

$$v_n=Tv_{n-1}\text{ and }w_n=Tw_{n-1},n=1,2,3\cdots,$$

then it follows from Theorem 1 that

$$v_0\ \leq\ v_1\ \leq\ \cdots\leq\ v_n\ \leq\ \cdots\leq\ w_n\ \leq\ \cdots\ \leq\ w_1\ \leq\ w_0,$$
$$\{v_n\}_{n=0}^{\infty}\text{ and }\{w_n\}_{n=0}^{\infty}\ \infty$$

converge to, respectively, v and $w\in[v_0,w_0]$, which are monotone solutions of the BVP(P). Moreover, for any $t\in(0,1]$, by Lemmas 2, we know that

$$
\begin{aligned}
(Tv_0)(t) &= \int_0^1 \left[G(t,s) + \frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau \right] a(s)f(s,0,0)ds \\
&> \int_0^1 G(t,s)a(s)f(s,0,0)ds \\
&\geq (t-\frac{t^2}{2})\int_0^1 (1-s)a(s)f(s,0,0)ds \\
&>0.
\end{aligned}
$$

So,

$$0 < (Tv_0)(t) \leq (Tv)(t) = v(t) \leq w(t), t \in (0,1],$$

which shows that v and w are positive solutions of the BVP (P).

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