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An Analytical Model for Regional Economic Convergence Study -- Based on DDE System

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Abstract

This paper proposed a reconstruction for regional economic convergence model from differential equations. By means of DDE technology, we introduced delayed logistic equation for population growth rate and found the evidence of oscillation in steady state. The result for short term is congruent with former researches while the occurrence of bifurcation in the long run leads to uncertainty. A threshold for time delay is essential to determine the disappearance of oscillation.

Keywords

Time Delay; Regional Convergence; Bifurcation.

1. Introduction

The conception of regional convergence aims to achieve balanced growth[21] and it explores whether a poorer region would catch up the richer one and if so, what affects the convergence speed. In Solow's theory[23], the poorer region usually have a higher growth rate due to decreasing marginal return of capital. Through more detailed division, the convergence can be divided to α, β or $club$ type. The α-convergence describes the gaps of income per capita among different regions as time increases[7]; and the β-convergence describes the speed of converging to a certain steady state[5,6,18]. The advantage of β-convergence is that the model considers heterogeneity and allows the comparison of regions with different initial endowments. The $club$-convergence investigates whether two regions with similar initial conditions would converge to the same level of output per capita[4].

Convergence study has been a significant topic for shrinking gap between rich and poor. By combining numerical and empirical test, Barro has found the evidence of convergence in US with speed 2% per year[4], and this is congruent with many other researches[1,12,13,18]. However, related studies have not yet discovered a unified perspective. Many researchers object the convergence theory and hold the idea that there exists none ubiquitous convergence around the world[19,25].

Most recent related researches are based on basic classical β-convergence model and derive results from empirical test. Obviously, the classical β-convergence model is easy to conduct empirics for its linearize equation. Nevertheless, the disadvantage of the model is evident—that the comparison of estimated results by empirical or numerical simulation of growth per capita exist an explicit deficiency-the convergence coefficient only depicts the speed to its own steady state, which in the long run , the fixed gap of rich and poor is still existing due to distinctive steady level. We reckon that it is one of the reasons that many researchers sometimes come to contradictory conclusions.

A constant population growth rate is mostly concerned in convergence studies. Nevertheless, population evolvement owns special and complex patterns and seems has its own evolving laws[24].

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Mustafa[20] held the idea that since population growth rate is time-lagged capital-dependent produce cycles. Guerrini et.al [15,16]constructed a time-to-build lagging system of capital and proved that Hopf bifurcation occurs with exogenous population growth rate; when logistic labor growth model was also introduced in, oscillating dynamics was founded at a certain time delay. Some other researches also confirmed the Hopf bifurcation occurs when the value of time delay crosses a certain value[2,10].Also, non-monotonic dynamics occur with two-time delays[17]. Related works verified sophisticated impact of time delayed population growth to capital growth trajectory. In our work, we are seeking for dynamics of economic convergence when considering a unique evolving path of population growth rate. Notably, the way we measure convergence is a little bit of different from former researches (See Part 2).

The rest of this paper consists of following parts. Part 2 deduces to an equation which will be used as convergence inspection method. Different with former researches, we take the natural log of output per capita as our agent for examining convergence. Part 3 explores the dynamics of Time-delay convergence model. Part 4 carries on numerical simulations and finally see part 5 for conclusion remarks.

2. Basic Modelling

Suppose the production function satisfies Cobb-Douglass form and k denotes effective capital per capita. $y = f(k)$ denotes effective production per capita. Follow by Solow's theory, the growth model of effective capital per capita takes the form:

$$\dot{\boldsymbol{k}} = \boldsymbol{s}\boldsymbol{f}(\boldsymbol{k}) - (\boldsymbol{n} + \boldsymbol{g} + \boldsymbol{\delta})\boldsymbol{k}, \boldsymbol{f}(\boldsymbol{k}) = \boldsymbol{k}^{\boldsymbol{\alpha}}, \mathbf{0} < \boldsymbol{\alpha} < \mathbf{1} \tag{1}$$

The equation satisfies that $f(0) = 0, f'(k) > 0, f''(k) < 0$ and Inada Condition. s, g, δ are exogenous fixed parameters denoting saving rate, technology progress and depreciation rate respectively. The Eq. (1) happens to be Bernolli type. Hence, the equation can be solved as:

$$\boldsymbol{k} = [\boldsymbol{v}_* + (\boldsymbol{v}_0 - \boldsymbol{v}_*)\boldsymbol{e}^{-(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{n}+\boldsymbol{g}+\boldsymbol{\delta})\boldsymbol{t}}]^{\frac{1}{1-\alpha}} \tag{2}$$

Where $v_t = k^{1-\alpha}$,the subscript '*' denotes steady state. In many former researches, $\beta = (1-\alpha)(n + g + \delta)$ represents the convergence parameter. We can easily get that $\frac{\partial \beta}{\partial n}$>0.

Apparently a constant n is the key to the solution of differential equation $\dot{k} = sf(k) - (n + g + \delta)k$.

Mankiw's theory is to derive a differential equation concerning Lny_t, Lny_0, $Lny_*$and β[18].

Since $\begin{cases} y_t = f(k) = k^{\alpha} \\ v_t = k^{1-\alpha} \end{cases}$, here we have:

$$\frac{\boldsymbol{dLny_t}}{\boldsymbol{dt}} = \frac{\boldsymbol{\alpha}}{\mathbf{1}-\boldsymbol{\alpha}}\frac{\boldsymbol{dLnv_t}}{\boldsymbol{dt}} = \boldsymbol{\beta}(\boldsymbol{Lny}_* - \boldsymbol{Lny_t}) \tag{3}$$

By solving the ordinary different Eq. (3) we have:

$$\boldsymbol{Lny_t} - \boldsymbol{Lny_0} = \boldsymbol{\beta}(\boldsymbol{Lny}_* - \boldsymbol{Lny_0}) \tag{4}$$

Lny_t denotes the log of production per capita, Lny_* denotes steady state. $\beta \in \mathbb{R}_+$ is convergence speed of a certain region converges to its steady point. However, it may not be that easy to derive the linear equation if n no longer keeps constant, especially n is a good

parameter to behave dynamical for that population growth rate usually has its own evolving trajectory. Admittedly, the Mankiw equation is perfect to conduct empirical test, the model itself has space to meliorate to enhance the explanatory power the reality .However, when considering dynamics in n,the equation shows different kinds of property.
Since $n = n(t)$, $\dot{k}$ is expressed as:

$$\dot{\boldsymbol{k}} = \boldsymbol{s}\boldsymbol{k}^{\boldsymbol{\alpha}} - (\boldsymbol{n}_{\boldsymbol{t}} + \boldsymbol{g} + \boldsymbol{\delta})\boldsymbol{k} \tag{5}$$

Supposing $v = k^{1-\alpha}$, for $\int_0^t \delta + g + \frac{\dot{L}}{L} du = (\delta + g)t + LnL_t$,by solving the differential equation yields:

$$\boldsymbol{v}_{\boldsymbol{t}} = \left(\boldsymbol{e}^{(\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{t}}\boldsymbol{L}_{\boldsymbol{t}}\right)^{\boldsymbol{\alpha}-\mathbf{1}}[\boldsymbol{v}_{\mathbf{0}} + (\mathbf{1}-\boldsymbol{\alpha})\boldsymbol{s}\int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{e}^{(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{u}}\boldsymbol{L}_{\boldsymbol{u}}^{\mathbf{1}-\boldsymbol{\alpha}}\boldsymbol{du}] \tag{6}$$

Also, we have $dLny_t/dt = \frac{\alpha}{1-\alpha} dLnv_t/dt$,$v_* = \lim_{t\to\infty} v_t = \frac{s}{n^*+\delta+g}$.
Taking natural logarithm on both sides of Eq. (6), it can be rewritten as:

$$\boldsymbol{Lnv}_{\boldsymbol{t}} = \boldsymbol{Ln}\left[\boldsymbol{v}_{\mathbf{0}} + (\mathbf{1}-\boldsymbol{\alpha})\boldsymbol{s}\int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{e}^{(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{u}}\boldsymbol{L}_{\boldsymbol{u}}^{\mathbf{1}-\boldsymbol{\alpha}}\boldsymbol{du}\right] - (\mathbf{1}-\boldsymbol{\alpha})[(\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{t} + \boldsymbol{LnL}_{\boldsymbol{t}}] \tag{7}$$

For equation (7), let's take derivative with respect to time t for both sides, then:

$$\frac{\boldsymbol{dLnv}_{\boldsymbol{t}}}{\boldsymbol{dt}} = \frac{,\boldsymbol{v}_*(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{n}+\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{e}^{(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{t}}\boldsymbol{L}_{\boldsymbol{t}}^{\mathbf{1}-\boldsymbol{\alpha}}}{\boldsymbol{v}_{\mathbf{0}}+(\mathbf{1}-\boldsymbol{\alpha})\boldsymbol{s}\int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{e}^{(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{\delta}+\boldsymbol{g})\boldsymbol{u}}\boldsymbol{L}_{\boldsymbol{u}}^{\mathbf{1}-\boldsymbol{\alpha}}\boldsymbol{du}} - (\mathbf{1}-\boldsymbol{\alpha})\left[(\boldsymbol{\delta}+\boldsymbol{g}) + \frac{\dot{\boldsymbol{L}}_{\boldsymbol{t}}}{\boldsymbol{L}_{\boldsymbol{t}}}\right] \tag{8}$$

Define $n_t = \frac{\dot{L}_t}{L_t}$ Hence Eq. (8) can be simplified as:

$$\frac{\boldsymbol{dLnv}_{\boldsymbol{t}}}{\boldsymbol{dt}} = \frac{\boldsymbol{v}_*(\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{n}_{\boldsymbol{t}}+\boldsymbol{\delta}+\boldsymbol{g})}{\boldsymbol{v}_{\boldsymbol{t}}} - (\mathbf{1}-\boldsymbol{\alpha})[(\boldsymbol{\delta}+\boldsymbol{g}) + \boldsymbol{n}_{\boldsymbol{t}}] \tag{9}$$

By substituting Eq. (3) to Eq. (9), we remark that:

$$\frac{\boldsymbol{dLny}_{\boldsymbol{t}}}{\boldsymbol{dt}} = \frac{\boldsymbol{\alpha}}{\mathbf{1}-\boldsymbol{\alpha}}\frac{\boldsymbol{dLnv}_{\boldsymbol{t}}}{\boldsymbol{dt}} = \boldsymbol{\alpha}\left[\frac{\boldsymbol{v}_*(\boldsymbol{n}_{\boldsymbol{t}}+\boldsymbol{\delta}+\boldsymbol{g})}{\boldsymbol{v}_{\boldsymbol{t}}} - (\boldsymbol{\delta}+\boldsymbol{g}+\boldsymbol{n}_{\boldsymbol{t}})\right] \tag{10}$$

When $v_t \to v_*$,according to Taylor expansion, since $dLny_t/dt = \frac{\alpha}{1-\alpha} dLnv_t/dt$,we get the linearized equation:

$$\frac{\boldsymbol{dLny}_{\boldsymbol{t}}}{\boldsymbol{dt}} = (\mathbf{1}-\boldsymbol{\alpha})(\boldsymbol{n}_{\boldsymbol{t}} + \boldsymbol{g} + \boldsymbol{\delta})(\boldsymbol{Lny}_* - \boldsymbol{Lny}_{\boldsymbol{t}}) \tag{11}$$

Unlike traditional β-convergence, we measure the speed of convergence by Eq. (11). It also describes the speed that a region converges to its steady state and considers heterogeneity among different economics. Specially, we suppose that population growth rate dynamics is in logistic form [24]. For simplicity, let's Set $p = Lny_t$. Henceforth, p denotes converging speed. The dynamics of logarithmic production per capita and population growth rate is given by the following autonomous ODE:

$$\dot{\boldsymbol{p}} = (\boldsymbol{1}-\boldsymbol{\alpha})(\boldsymbol{n_t}+\boldsymbol{g}+\boldsymbol{\delta})(\boldsymbol{p}^*-\boldsymbol{p}) \tag{12}$$

$$\dot{\boldsymbol{n}} = \boldsymbol{n}(\boldsymbol{a}-\boldsymbol{bn}) \tag{13}$$

p^*denotes steady state. to study the property of the system in equilibrium point, by setting $\dot{p} = \dot{n} = 0$,for all t,we have one nontrivial stable points: $(p_1, n_1) = (p^*, \frac{a}{b})$. For any $a, b, p^*, g, \delta \in \mathbb{R}_+$, $\alpha \in (0,1)$, $a, n \neq 0$,we find that the first stable point$(p_1, n_1)$is in first quadrant.
By applying first order Taylor expansion around (p_1, n_1),we get the linearized systems:

$$\mathcal{System}\ 1\text{:} \begin{bmatrix}\dot{p}\\ \dot{n}\end{bmatrix} = \begin{bmatrix}-(1-\alpha)(\frac{a}{b}+g+\delta) & 0\\ 0 & -a\end{bmatrix}\begin{bmatrix}p-p_1\\ n-n_1\end{bmatrix}$$

The characteristic equation is given by:

$$\left[\lambda+(1-\alpha)\left(\frac{a}{b}+g+\delta\right)\right](\lambda+a) = 0$$

The two eigenvalues $\lambda_{1,2} \in \mathbb{R}$ are both negative. Hence, the dynamics of $\mathcal{System}$ 1 does not admit any bifurcations.

3. Modelling Time-delay

Let's consider that after a time period τ, $(\tau > 0)$, p still exert influence on $\dot{p}$.Hence, the dynamics of $\dot{p}$ depends on current state $p(t)$ and $p(t-\tau)$ with coefficient $\varepsilon_1, \varepsilon_2$ respectively. We assume population growth rate n evolves over time delayed $n(t-\tau)$(See [11,15,16] for labor delayed logistic system). The DDE system is as followed:

$$\dot{\boldsymbol{p}} = (\boldsymbol{1}-\boldsymbol{\alpha})(\boldsymbol{n}+\boldsymbol{g}+\boldsymbol{\delta})(\boldsymbol{p}^*-\boldsymbol{\varepsilon_1 p}-\boldsymbol{\varepsilon_2 p_d}) \tag{14}$$

$$\dot{\boldsymbol{n}} = \boldsymbol{n}(\boldsymbol{a}-\boldsymbol{bn_d}) \tag{15}$$

For simplicity, $p(t-\tau) = p_d, n(t-\tau) = n_d$.
Corollary 1:Since $\varepsilon_1+\varepsilon_2 = 1$,the steady point is same as (p_1, n_1).
Proof: Since $\dot{p} = \dot{n} = 0, n = n_d$,we can get $p = p_d = p^*/(\varepsilon_1+\varepsilon_2)$.
Remark 1: Eqs. (14,15) has a non-zero equilibrium(p_2, n_2)= $(\frac{p^*}{\varepsilon_1+\varepsilon_2}, \frac{a}{b})$.
To analyze local property around non-zero equilibrium, we get two Jacobian matrix around (p_2, n_2):

$$\mathcal{J}_a = \begin{bmatrix}-\varepsilon_1(1-\alpha)(g+\delta+\frac{a}{b}) & 0\\ 0 & 0\end{bmatrix} \quad \mathcal{J}_b = \begin{bmatrix}-\varepsilon_2(1-\alpha)(g+\delta+\frac{a}{b}) & 0\\ 0 & -a\end{bmatrix}$$

The characteristic equation $D(\lambda, \tau)$ satisfies:

$$\boldsymbol{D}(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \boldsymbol{DET}\left(-\boldsymbol{\lambda E}+\boldsymbol{\mathcal{J}_a}+\boldsymbol{\mathcal{J}_b}\boldsymbol{e}^{-\boldsymbol{\lambda\tau}}\right) = \boldsymbol{0} \tag{16}$$

Where E is 2×2 identity matrix. Eq. (16) is equivalent to:

$$\boldsymbol{H_1(\lambda,\tau)\cdot H_2(\lambda,\tau)=0} \tag{17}$$

By solving Eq. (17), we can derive that:

$$\boldsymbol{H_1(\lambda,\tau)=\lambda+(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1+(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_2e^{-\lambda\tau}} \tag{18}$$

$$\boldsymbol{H_2(\lambda,\tau)=\lambda+ae^{-\lambda\tau}} \tag{19}$$

On the other hand, it is equivalent to study the property when $H_1(\lambda,\tau)=0$(or $H_1(\lambda,\tau)=0$).

Remark 2: If time delay $\tau=0$,the system is same as $\mathcal{System}\ 1$,that the two negative eigenvalues admit one uniformly stable node.

Since Eqs. (18,19) is a nondegenerate system, $\lambda=0$ is not a characteristic root. Hence, the key to the solutions of Eqs. (18,19) is the occurrence of pure imaginary root of time delay τ. Suppose $\lambda=i\omega,\omega\in\mathbb{R}_+$.With the increase of τ,the pure imaginary root would cross the imaginary axis at a threshold τ_*. For convenience of calculation, we rewrite equation (18,19) as following:

$$\mathcal{System}\ 2:\quad \begin{aligned} H_1(\lambda,\tau)&={P_{RE}}^1+{P_{IM}}^1+({Q_{RE}}^1+{Q_{IM}}^1)e^{-\lambda\tau}\\ H_2(\lambda,\tau)&={P_{RE}}^2+{P_{IM}}^2+({Q_{RE}}^2+{Q_{IM}}^2)e^{-\lambda\tau}\end{aligned}$$

Where $P^j={P_{RE}}^j+{P_{IM}}^j, Q^j={Q_{RE}}^j+{Q_{IM}}^j,(j=1,2)$. For $H_1(\lambda,\tau)$, ${P_{RE}}^1=(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1$, ${P_{IM}}^1=i\omega_1$, ${Q_{RE}}^1=(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_2$, ${Q_{IM}}^1=0$ .Similarly, for $H_2(\lambda,\tau)$, ${P_{RE}}^2={Q_{IM}}^2=0$, ${P_{IM}}^2=i\omega_2$ and $Y_{RE}=a$ .Then, $\omega_j,(j=1,2)$ satisfies the following constraint equations(See Beretta[9] for more detailed operation):

$$sin\omega_j\tau=\frac{D_1^j}{|Q^j|^2}(j=1,2) \tag{20}$$

$$\boldsymbol{cos\omega_j\tau=-\frac{D_2^j}{|Q^j|^2}(j=1,2)} \tag{21}$$

Where $|Q^j|$ represents the norm of the complex; D_1^j,D_2^j denote the value of determinants $\begin{vmatrix}{P_{IM}}^j & {P_{RE}}^j\\ {Q_{IM}}^j & {Q_{RE}}^j\end{vmatrix}$, $\begin{vmatrix}{P_{IM}}^j & -{P_{RE}}^j\\ {Q_{RE}}^j & {Q_{IM}}^j\end{vmatrix}$ respectively.

Lemma 1: With $\tau=\tau_k^j\ (k=0,1,2\ldots)$, the pure imaginary eigenvalues λ_j and time delay τ^j for $H_1(\lambda_1,\tau)$, $H_2(\lambda_2,\tau)$ are:

$$\boldsymbol{\lambda_1=-(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1\tan\left[\mathbf{arccos}-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right]i,(\mathbf{resp.}\,\lambda_2=ai)} \tag{22}$$

$$\boldsymbol{\tau^1=\frac{1}{\omega_1}\left[\mathbf{arccos}\left(-\frac{\varepsilon_1}{\varepsilon_2}\right)+2k\pi\right],(\mathbf{resp.}\,\tau^2=\frac{1}{\omega_2}(\frac{\pi}{2}+2k\pi))} \tag{23}$$

Proof: It's easy to derive $cos\omega_1\tau = -\frac{\varepsilon_1}{\varepsilon_2}$, $sin\omega_1\tau = \frac{\omega}{(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_2}$ from Eq. (22).By doing division, we get $tan\omega_1\tau = -\frac{\omega}{(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1}$.Considering $\omega_1\tau = \arccos\left(-\frac{\varepsilon_1}{\varepsilon_2}\right)$, ω_1 thus equals to: $-(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1 \tan\left[\arccos-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right] > 0$. Similarly, by applying same method to H_2,we thus have $cos\omega_2\tau = 0$ and $sin\omega_2\tau = \omega_2/a$.

Proposition 1: A pair of pure imaginary roots for $H_1(\lambda_1,\tau)$ only occurs when following statement both satisfied:

i. $\varepsilon_1/\varepsilon_2 \leq 1$,for any $\varepsilon_1, \varepsilon_2 \in (0,1)$;

ii. $\omega_1 \leq (1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_2$,for any $\omega_1 \in \mathbb{R}_+$.

Proposition 2: Since $a \neq -(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1 \tan\left[\arccos-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right]$,the system undergoes bifurcation when $\tau^j = \tau_*^j$,with a pair of pure conjugate imaginary roots $\lambda_j = \pm i\omega_j$.

Corollary 2: $a = -(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1 \tan\left[\arccos-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right]$ will not occur.

Proof: If $a = -(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1 \tan\left[\arccos-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right]$, Eq. (11) has double-nonzero eigenvalue, thus results in degeneration of the system.

Proposition 3: As τ^j increases, the pairs of pure conjugate imaginary roots $\lambda_j = \pm i\omega_j$ would go through the imaginary axis from left to right.

Proof: The direction of crossing depends on the following transversality condition:

i Since $Sign\left\{Re\left(\frac{d\tau}{d\lambda}\right)\right\}_{\tau=\tau_*} > 0$,the pure imaginary roots λ cross from left to right;

ii Since $Sign\left\{Re\left(\frac{d\tau}{d\lambda}\right)\right\}_{\tau=\tau_*} < 0$,the pure imaginary roots λ cross from right to left.

Set λ_2 as an example. Taking the partial derivative with respect to τ for both side of $H_2(\lambda,\tau) = 0$ and doing transposition, we obtain:

$$\left.\frac{d\tau^2}{d\lambda_2}\right|_{\lambda_2=\omega_2 i} = -\frac{1}{{\lambda_2}^2} - \frac{\tau^2}{\lambda_2} \tag{24}$$

By simplifying the above differential equation, it follows that:

$$\left.Sign\left\{Re\left(\frac{d\tau}{d\lambda_2}\right)\right\}\right|_{\lambda_2=\omega_2 i} = Sign\left\{\frac{1}{\omega_2^2}\right\} \tag{25}$$

Where $Sign$ is sign function. For any $\omega, \sigma, \tau \in \mathbb{R}_+$, Eq. (25) is positive.

The stable switch depends on the lesser value of τ^1, τ^2 when $H_1 = 0(or\ H_2 = 0)$.Let $\tau_*^{min} = min(\tau^1, \tau^2)$. τ_*^{min} is a demarcation point. Obviously, τ_*^{min} occurs when $k = 0$.Thus we derive that:

$$\tau_*^{min} = min\left(\frac{\arccos\left(-\frac{\varepsilon_1}{\varepsilon_2}\right)}{-(1-\alpha)\left(g+\delta+\frac{a}{b}\right)\varepsilon_1 \tan\left[\arccos-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right]}, \frac{\pi}{2a}\right) \tag{26}$$

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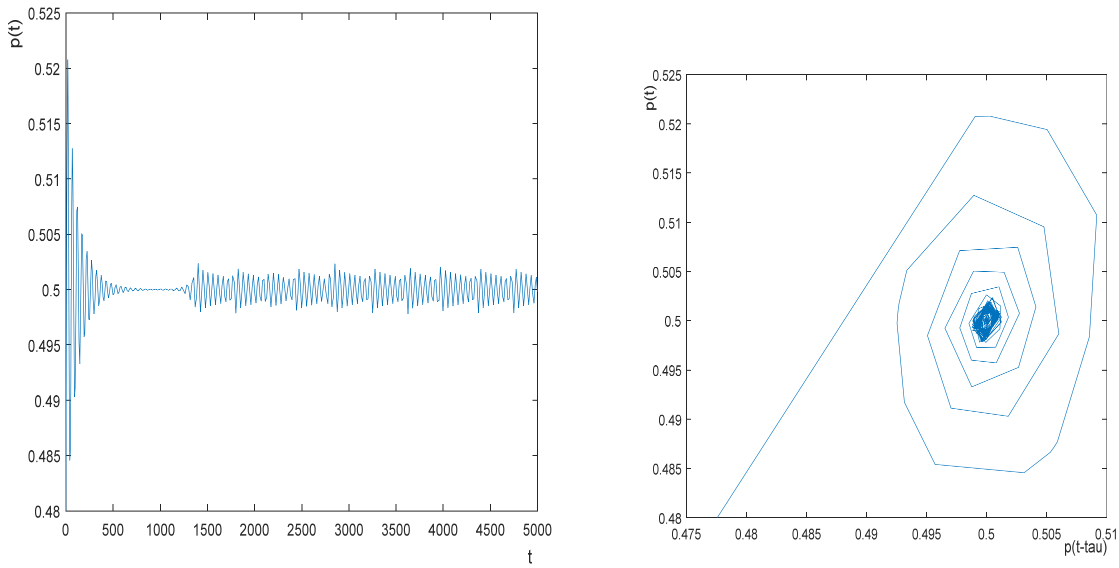
4. Numerical Simulation

Firstly, since we notice that $\frac{\varepsilon_1}{\varepsilon_2} > 1$, ω_1 is a complex with non-trivivial real part. That indicates we cannot derive a real positive time delay τ^1 and the bifurcation threshold is unsolvable.
For numerical simulation, let's consider $\frac{\varepsilon_1}{\varepsilon_2} \leq 1$.That indicates the equation $H_1(\lambda, \tau) = 0$ with a pair of pure imaginary roots is explicit. we set:

$$A = 1 \quad \delta = 0.1 \quad \alpha = 0.45 \quad s = 0.8$$
$$a = 0.08 \quad b = 0.92 \quad g = 0.3$$
$$\varepsilon_1 = 0.47 \quad \varepsilon_2 = 0.53 \quad p^* = 0.5$$

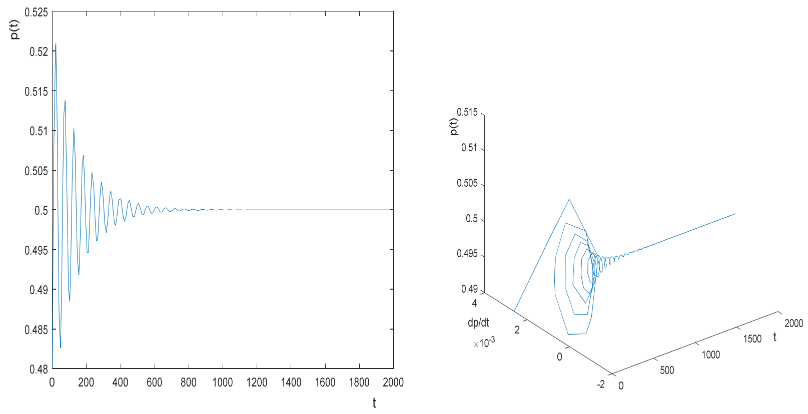
For the above parameter intercalation, the uniform stable node is $(p^*, n^*) = (0.5, 0.087)$ and threshold of bifurcation $\tau_0^1 \approx 40.557, \tau_{(0)}^2 \approx 19.635$.When $\tau \in [0, \tau_{(0)}^2)$, $H_2(\lambda, \tau)$ is stable and by means of simulations we can determine the property of cross direction of $\lambda_1 = \omega_1 i$.
Figure 1 (a) and (b) display the dynamics of unstable equilibrium for $p(t)$.As time t increases, $p(t)$ oscillates with relatively stable amplitude at $t \approx 1300$.



(a) (b)

Figure 1. (a):Time evolution for natural log of output per capita $p(t)$,at $\tau = 18.6$.
(b): Trajectory chart for variable $\dot{p}(t)$ and $p(t)$,at $\tau = 18.6$.

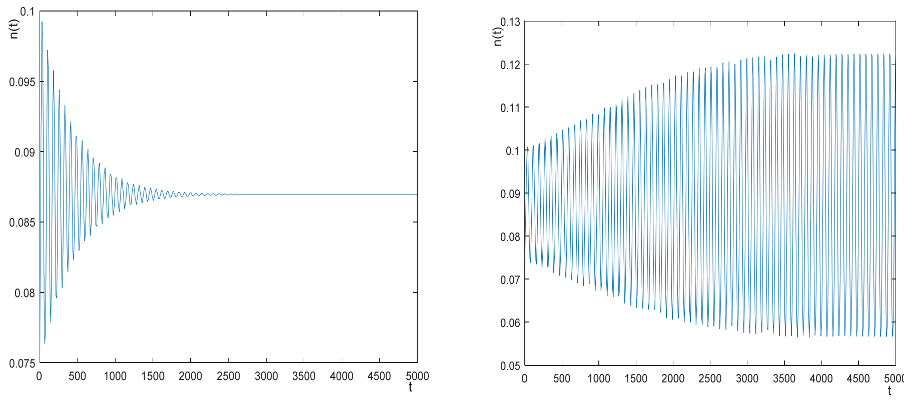
Compare Figure 1 (a)&(b) with Figure 2 (c)&(d) we can find that when time delay τ evolves across the significant value $\tau_{(0)}^2$,the system converges to stable point $p^* = 0.5$. Figure 2 (d) exhibit the trajectory to the stable state in three-dimensional axis. Hence, the bifurcation switch for $H_1(\lambda, \tau)$ is from unstable status to non-trivival stable equilibrium.



(c) (d)

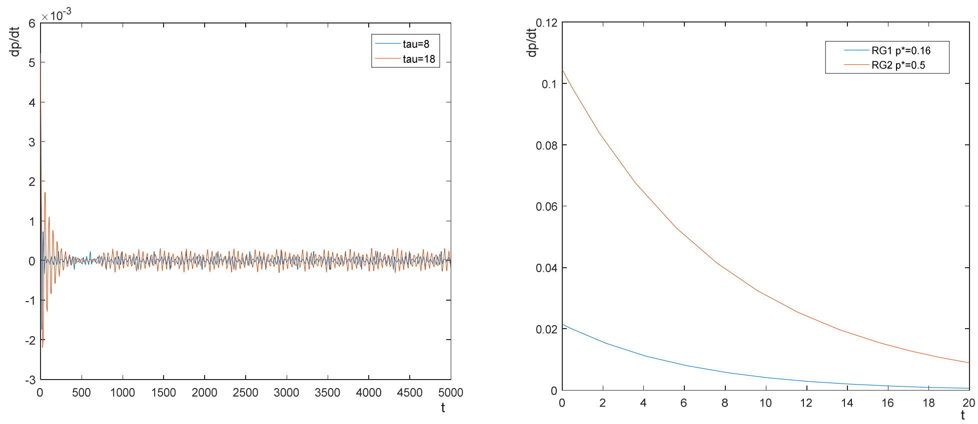
Figure 2. (c): Time evolution for natural log of output per capita $p(t)$,at $\tau = 20$. (d)The dynamics of $\dot{p}(t)$, $p(t)$ as time t evolves in three-dimensional coordinates, at $\tau = 20$.

The results for $H_2(\lambda, \tau)$ shown in Figure 3(e) and (f) are congruent with what we have drawn before. When $\tau \in [0, \tau_{(0)}^2)$, $H_2(\lambda, \tau)$ is uniformly stable and once crosses the threshold, the bifurcation happens.



(e) (f)

Figure 3. (e): Time evolution for population growth rate $n(t)$,at $\tau = 18.6$. (f): Time evolution for population growth rate $n(t)$,at $\tau = 20$.



(g) (h)

Figure 4. (g): Time evolution for GDP per capita growth rate dp/dt ,at $\tau = 8,18$,respectively. (h): Time evolution for population growth rate dp/dt ,at $\tau = 20$,$p^* = 0.16,0.5$,respectively.

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5. Conclusion

This paper adopts a new approach to analyze regional economic convergence measuring by the natural logarithm of GDP per capita. We use differential equation to depict steady-state-related growth model. In the first part, we conclude that bifurcation does not occur without time delay. Thus, we can compare the value of $\dot{p}_i(t)$ in region i to examine the developing speed of converging to its steady state. We considered heterogeneity among different economies, which reflected by saving rate s ,capita-labor contribution coefficient α , depreciation rate δ,technological progress rate g and different logistical coefficients for n.The heterogenous parameters determine distinctive steady value p^* so that the comparison of growth for GDP per capita among different areas with diverse endowment has more practical significance.

In the second part, we introduced DDE systems into the dynamical analysis. The system is identical to former one when the coefficient for time-lagged term is naught. The introduction of the time-lagged term makes the dynamics undergoes a more complicated behavior. The phase space turns to an infinite Banach space, which brings about boundless characteristic roots. The existance of a pair of pure imaginary roots implies that the alter of stable status and Hopf bifurcation. In the unstable range of time delay, different τ result in different oscillation of $\dot{p}(t)$, hence, the forecast or comparison of a series of regions exist uncertainty to some extent. As is shown in picture (g), by setting different parameters of time delay, the vibration of $\dot{p}(t)$ varies. However, it is hard to forecast the evolution trends of amplitude as τ increases and it brings us some challenges when facing different time delay or initial status.

Our work concludes that bifurcation for $\dot{p}(t)$ occurs when $\tau < \tau_*^{min}$.The results enlightens us that if one region's growth rate of GDP per capita is related to its recent data, the bifurcation is more likely to appear. With a small τ,the growth of GDP per capita in an economy affected by its recent parameter, thus indicates that it is an emerging economy and there exists a measure of uncertainty and stochasticity.

For the region who obsesses a larger τ,the equilibrium point is stable and economy would converge to the point. In this case, we compare growth rate among different regions. By setting different parameters, we derive that $p^* = 0.16$ for region 1 and $p^* = 0.5$ possesses different growth rate under same initial iteration point. To exhibit details we amplified the pic and set time range from 0~20.We find that a region with lower steady state owns a smaller growth rate. This result indicates that a poorer region would never catch up the richer one, in other words, the gap is widening. Unless an exogenous shock (e.g., preferential policy, technology spillover, etc.) changes its own parameters. This kind of change could be steady state's variation(e.g. acquire a higher p^*), or time delay changes from stable to bifurcation, or labor's rise in short time, etc. The results also holds the idea that one region with abundant endowment usually own a higher p^* and growth potential [8,14]. On the contrary, it is hard for a region with wide gap to catch up. However, unlike former researches, in the long run at this case, the $\dot{p}(t)$ would oscillate with decreasing amplitude until the system is stable.

At last, we compare two regions with same steady state but different initial parameter. Since the effect of bifurcation occurs with a big t(we set time range up to 2,000 or 5,000 in Figure (a)(c)(d)(e)(f) their initial growth is same roughly, regardless of whether bifurcation occurs. That implies two poor regions would convergent at first, but in the long run, oscillation makes the trend unpredictable. Besides, with time increases, exogenous shock also makes the long-run prediction unlikely precise.

This paper also exists some deficiencies. It is hard to get data for thousands of years (Usually the unit for t is year), so it is not possible to verify by empirical analysis of long-run behavior of our work. The dynamics of muliti-time delays model is still unexplored in this study. Same as Matsumoto et al. [3],one drawback of the model is that the trajectories of the curve could not keep in first quadrant. Also, due to stickiness in macroeconomy, parameters would not vary smoothly as in our equations. Looking forward to further explorations.

Declaration of Competing Interest

None.

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